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Magyari, E.; Pop, I.; Keller, B.

A Note on the Free Convection from Curved Surfaces

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1. Introduction

The self-similar free convection from the outer surface of a heated body is of great interest in several technical and environmental heat transfer processes that occur in practice, such as in meteorological devices, builing insulation systems, heat film sensors, energy storage in enclosures, etc. For a comprehensive review of this topic see e.g. Gebhart et al. [1], Gersten and Herwig [2], Bejan [3], and Schlichting and Gersten [4]. While the bulk of the classical research was concerned with simple geometries like flat plates, cylinders, and spheres, the general case of the free convective flow over a nonisothermal two-dimensional body of arbitrary geometric configuration has been attacked only a couple of years ago by Pop and Takhar [5]. The similar problem for nonisothermal curved surfaces embedded in fluid-saturated porous media has been discussed by Nakayama et al. [6–8] and for the case of micropolar fluids by Char and Chang [9]. In both of these problems, the existence of a family of curved surfaces and of corresponding temperature distributions which permit similarity solutions of power-law type has been proven [5–9]. The equation of the corresponding shape curves has been given in [5] and [6] in terms of a series expansion which only converges in the range $0 < n \leq 1/2$ of the shape exponent n.

The aim of the present note is to show that (i) the two-dimensional curved surfaces which allow for self-similar free convection flows exist for any $n > 0$; (ii) their equation may be expressed in terms of Gauss' hypergeometric function; and (iii) to discuss the main features of these surfaces as functions of n.

2. Basic equations

In the notation of Pop and Takhar [5] the parametric equations $\{z = z(x), r = r(x)\}$ of the shape curve are given by

$$z(x) = \int_0^x (g_x/g)\, dx' \,, \qquad r(x) = \int_0^x \left[1 - (g_x/g)^2\right]^{1/2} dx' \tag{1}$$

where x denotes the boundary layer coordinate along the curved surface while r and z are the horizontal and the vertical coordinates of the points of the shape curve $z = z(r)$, respectively. The origin of the coordinate system $(r,\ z)$ lies on the lower stagnation line of the heated surface and g_x denotes the tangential component of the acceleration due to the gravity g. For the similarity of power-law type, we assume that g_x has the following form:

$$g_x = g \cdot \left(\frac{x}{x_r}\right)^n \tag{2}$$

where $n > 0$ ($n = 0$ corresponds to a vertical plane) and x_r is a reference length measured along the curved surface; x_r represents also the maximum allowed value of the boundary layer coordinate $x : 0 \leq x \leq x_{\max} \equiv x_r$. The first integral in (1) is immediate and the second one has been given in the earlier literature in terms of a series expansion which, as pointed out in [5] and [6], it converges only for $0 < n \leq 1/2$.

3. General solution

The second integral in (1) can be reduced easily to an incomplete beta function which in turn may be expressed in terms of Gauss' hypergeometric function $F(a, b; c; \xi)$ (see e.g. [10]). The results of these calculations, which is valid for any $n > 0$, reads

$$\frac{z}{z_r} = \left(\frac{x}{x_r}\right)^{n+1}, \qquad \frac{r}{z_r} = (n+1)\, \frac{x}{x_r}\, F\left(\frac{1}{2n}, -\frac{1}{2}; \frac{1}{2n} + 1; \left(\frac{x}{x_r}\right)^{2n}\right). \tag{3}$$

The variation range of z and r is $0 \leq z \leq z_{\max} = z|_{x=x_r} \equiv z_r = x_r/(1+n)$, and $0 \leq r \leq r_{\max} = r|_{x=x_r}$, respectively. The slope dz/dr of the shape curves $z = z(r)$ is always zero in the origin $x = r = z = 0$, whereas at the upper end $r = r_{\max}$ of the r-range, one has $(dz/dr)\,|_{r=r_{\max}} = \infty$ for any $n > 0$. Hence, all the shape curves $z = z(r)$ end in the point

$(r, z) = (r_{\max}, z_r)$ with a vertical tangent, where

$$\frac{r_{\max}}{z_r} = (n+1) \cdot F\left(\frac{1}{2n}, -\frac{1}{2}; \frac{1}{2n}+1; 1\right) = \frac{\Gamma\left(\frac{3}{2}\right) \cdot \Gamma\left(\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2n}+\frac{1}{2}\right)}, \tag{4}$$

Γ being the gamma function. (Here the known relation $F(a, b; c; 1) = \Gamma(c)\,\Gamma(c-a-b)\,\Gamma^{-1}(c-a)\,\Gamma^{-1}(c-b)$ has been used, see [10].) Therefore, the equation of the shape curves in its implicit form, $r = r(z)$, reads

$$\frac{r}{r_{\max}} = \frac{F\left(\frac{1}{2n}, -\frac{1}{2}; \frac{1}{2n}+1; \zeta\right)}{F\left(\frac{1}{2n}, -\frac{1}{2}; \frac{1}{2n}+1; 1\right)}\, \zeta^{1/2n} \quad \text{where} \quad \zeta \equiv \left(\frac{z}{z_r}\right)^{2n/n+1}. \tag{5}$$

All the above equations are valid for any $n > 0$. However, from (5) one obtains for $n \gg 1$ the following asymptotic relations:

$$\frac{z}{z_r} \cong \left(\frac{r}{n z_r}\right)^n, \qquad \frac{r_{\max}}{z_r} \cong n, \qquad n \gg 1. \tag{6}$$

As an illustration, Fig. 1 shows the shape curves for different values of n. The main difference between the curves corresponding to $n \le 1$ and $n \gg 1$ consists of the large and flat bottom part of the latter ones in accordance with the relations given by (6). This finding is also supported by the behavior of the curvature C of the shape curves $z = z(r)$ in the origin of the coordinate system (r, z). Indeed, from (3) one obtains

$$C = \left.\frac{z''}{(1+z'^2)^{3/2}}\right|_{r=0} = \frac{n}{(n+1)\,(n^2+2n+2)^{3/2}} \lim_{x\to 0}\left(\frac{x}{x_r}\right)^{n-1} \tag{7}$$

where primes denote differentiation with respect to x. Expression (7) which shows that $C = \infty$ for $0 < n < 1, C = 1/2$ for $n = 1$, while for any $n > 1, C = 0$. On the other hand, from (5) we get the following elementary solution for $n = 1$:

$$\frac{r}{z_r} = \frac{\pi}{2}\frac{r}{r_{\max}} = \sqrt{\frac{z}{z_r}\left(1-\frac{z}{z_r}\right)} + \arcsin\sqrt{\frac{z}{z_r}}. \tag{8}$$

Having in mind the properties of the hypergeometric functions, [10], the general solution given by (5) may be written in the (equivalent) form

$$\frac{r}{r_{\max}} = 1 - \frac{F\left(1-\frac{1}{2n}, \frac{3}{2}; \frac{5}{2}; 1-\zeta\right)}{F\left(1-\frac{1}{2n}, \frac{3}{2}; \frac{5}{2}; 1\right)}\,(1-\zeta)^{3/2}. \tag{9}$$

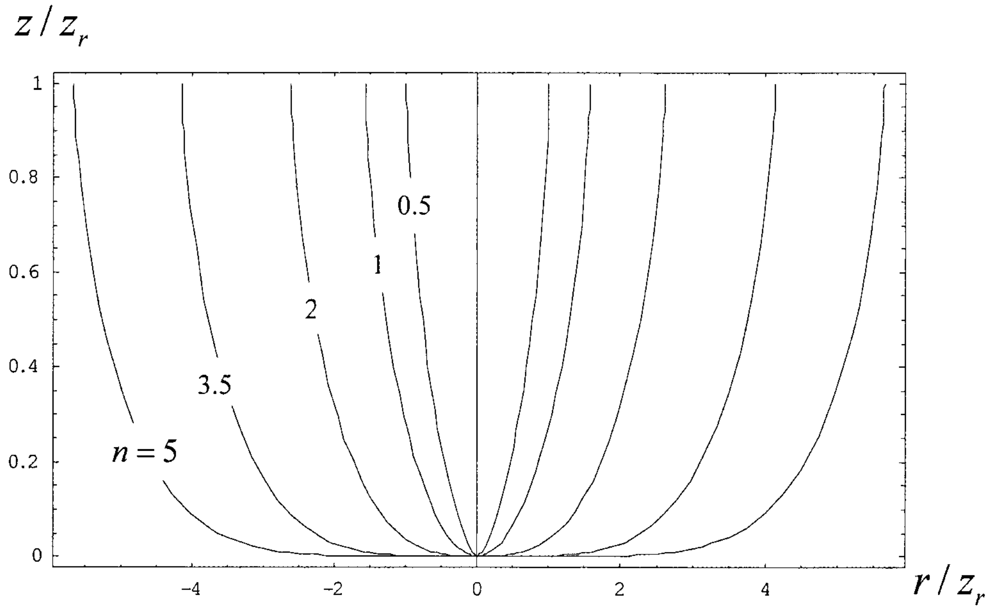

Fig. 1. Shape curves for $n = 0.5$, 1.0, 2.0, 3.5, and 5.0. The variation range of r/z_r is $0 \le r \le r_{\max}/z_r$, where $r_{\max}/z_r$ is given by eq. (4). The large and flat bottom parts of the curves corresponding to $n \gg 1$ (and described by eq. (6)), can already be seen for $n = 3.5$ and 5.

This solution has the advantage that in the particular cases $(2n)^{-1} - 1 \equiv N = 0, 1, 2, \ldots$, i.e., for $n = 1/2,\ 1/4,\ 1/6, \ldots$ the function F in the numerator of eq. (9) reduces to a polynomial of degree N in $1 - \zeta$:

$$F\left(1 - \frac{1}{2n}, \frac{3}{2}; \frac{5}{2}; 1 - \zeta\right) = \sum_{k=0}^{N} \frac{(-N)_k\,(3/2)_k}{(5/2)_k} \frac{(1-\zeta)^k}{k!} \tag{10}$$

where $(a)_k$ stands for Pochhammer's symbol, $(a)_k = a(a+1)\,(a+2)\ldots(a+k-1)$, $(a)_0 = 1$, see [10].

For $0 < n \le 1/2$, eq. (9) is identical to that with the results reported in [5] and [6] as obtained by a series expansion of the integrand in eq. (1), see [6]. For the values $n = 1/2,\ 1/4,\ 1/6, \ldots$ in this range, one immediately obtains from eq. (9) the special solutions reported in [6]. In particular, for $n = 1/2$, i.e. $N = 0$, we obtain $r_{\max} = z_r = 2x_r/3$, $\zeta = (z/z_r)^{2/3}$, and

$$\frac{r}{r_{\max}} = \frac{r}{z_r} = 1 - \left(1 - \zeta^{2/3}\right)^{3/2} = 1 - \left[1 - \left(\frac{z}{z_r}\right)^{2/3}\right]^{3/2}. \tag{11}$$

In this case the explicit equation $z = z(r)$ can be also written as

$$\frac{z}{z_r} = \left[1 - \left(1 - \frac{r}{r_{\max}}\right)^{2/3}\right]^{3/2}. \tag{12}$$

4. Summary

In the present research note the problem of the heated two-dimensional curved surfaces which can give rise to self-similar free convection flows (both in a quiescent fluid and in a fluid-saturated porous medium) has been reconsidered. The equation of the shape curves, valid for any positive value of the shape exponent n, was given in terms of Gauss' hypergeometric function. In the range $0 < n \le 1/2$ a whole agreement with the results of earlier authors, [5–9], has been found. However, for $n = 1$ the equation of the shape curve has been expressed in terms of elementary functions. The curvature C of the surface at the lower stagnation line (the origin of the coordinate system) is ∞ for $0 < n < 1$, $1/2$ for $n = 1$, while for any $n > 1$, $C = 0$ holds. In the range $n \gg 1$ of the shape exponent, the ratio $r_{\max}/z_{\max}$ of the horizontal and vertical dimensions of the curved surface increases with n according to $r_{\max}/z_{\max} \cong n$. Thus, the limiting case $n \to \infty$ corresponds to a horizontal plane of which hot side is faced downwards. Below this infinite surface no free convection can occur. It is worth mentioning however that if this horizontal surface would reduce to a plate of finite dimensions, the fluid could escape by spilling over its edges (see e.g. Bejan [3], p. 196).

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Addresses: Dr. E. Magyari, Dr. B. Keller, Chair of Physics of Buildings, Institute of Building Technology, Swiss Federal Institute of Technology (ETH) Zürich, CH-8093 Zürich, Switzerland; Prof. I. Pop, Faculty of Mathematics, University of Cluj, RO-3400 Cluj, CP 253, Romania