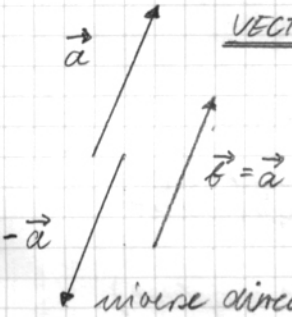
- VECTORS basics
- LINES distance, intersection
- PLANES distance, intersection

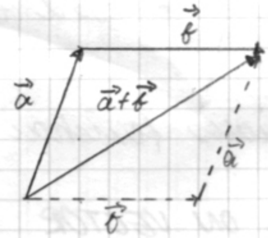


VECTOR: • length $|\vec{a}|$
• direction

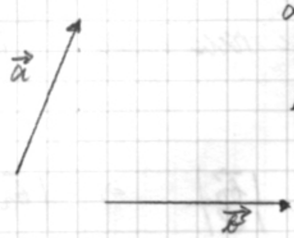
vectors can be moved around in space

inverse direction

ADD.

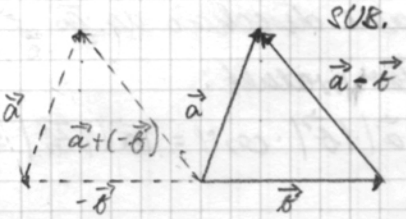


$\vec{a}+\vec{b} = \vec{b}+\vec{a}$ commutative

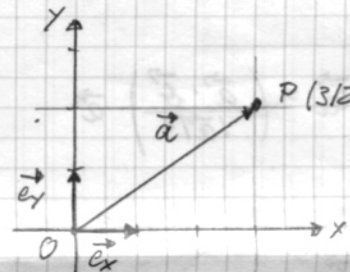


scalar MULT.

SUB.



calculations as in $\mathbb{R}$!



$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

base vectors $\Rightarrow$ $\vec{a} = 3\cdot\vec{e_X} + 2\cdot\vec{e_Y}$

calculation ⟵

short: $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\Rightarrow$ every point in space can be identified with the $\overrightarrow{OP}$

$\Rightarrow$ $\vec{a}+\vec{b} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix}$

$\vec{a}-\vec{b} = \begin{pmatrix} a_1-b_1 \\ a_2-b_2 \end{pmatrix}$

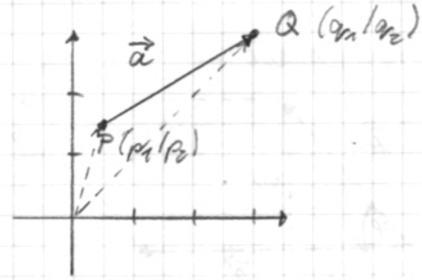
$\lambda\vec{a} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix}$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$

$\vec{a} = \vec{b}$ iff $a_1 = b_1$ & $a_2 = b_2$

$\vec{a} = \overrightarrow{OQ} - \overrightarrow{OP}$

$\Rightarrow$ $\vec{a} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \end{pmatrix}$



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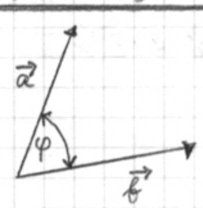
SCALAR PRODUCT $\vec{a}\cdot\vec{b}$ ⇝ angle between two vectors

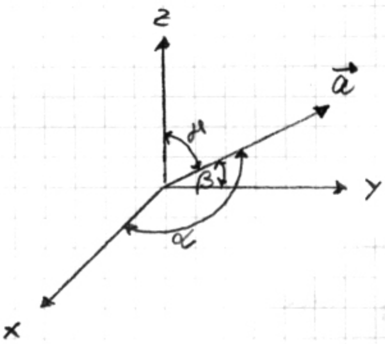


$\vec{a}\cdot\vec{b} := |\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi$ with $0 \le \varphi \le 180°$ (enclosed angle)

- $\vec{a} \perp \vec{b}$ iff. $\cos\varphi = 0$ iff $\vec{a}\cdot\vec{b} = 0$
- $\vec{a}\cdot\vec{b} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1\cdot b_1 + a_2\cdot b_2$
- $\varphi = \arccos \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

up to now everything in $\mathbb{R}^2$, but also valid in $\mathbb{R}^3$

ANGLE OF DIRECTION: $\rightsquigarrow$ angle between vector an base vectors
together with $|\vec{a}|$ this gives a unique description of $\vec{a}$



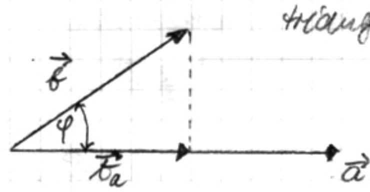
- $\cos\alpha = \dfrac{\vec{a}\cdot\vec{e_x}}{|\vec{a}||\vec{e_x}|} = \dfrac{a_1}{|\vec{a}|}$

  $\cos\beta = \dfrac{a_2}{|\vec{a}|} \qquad \cos\gamma = \dfrac{a_3}{|\vec{a}|}$

- $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, i.e. the angle are interdependent

$\Rightarrow$ spherical coordinates function this way ($|\vec{a}|=1$ as point on sphere)

PROJECTION OF VECTOR ON VECTOR



triangle: $\cos\varphi = \dfrac{|\vec{b_a}|}{|\vec{b}|} \Rightarrow |\vec{b_a}| = |\vec{b}|\cdot\cos\varphi$

$\vec{b_a}$ has same direction as $\vec{a} \Rightarrow \vec{b_a} = |\vec{b_a}|\cdot\dfrac{\vec{a}}{|\vec{a}|}$

with scalar product:

$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\varphi = |\vec{a}||\vec{b_a}| \Rightarrow |\vec{b_a}| = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$\Rightarrow \vec{b_a} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\cdot\vec{a}$

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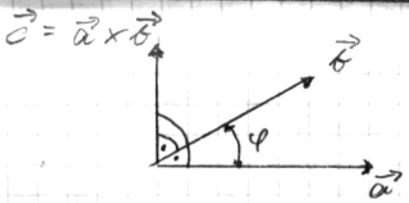
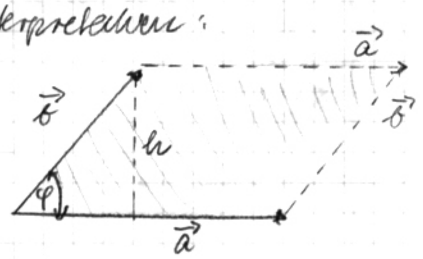
VECTOR PRODUCT (cross product) $\vec{a}\times\vec{b}$ (exists only in $\mathbb{R}^3$!)



let $\vec{c} := \vec{a}\times\vec{b}$ then $\vec{c}$ is defined by:

- $\vec{c}\perp\vec{a}$ & $\vec{c}\perp\vec{b}$ that is $\vec{a}\cdot\vec{c} = 0 = \vec{b}\cdot\vec{c}$
- $|\vec{c}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\varphi$ with $0 \le \varphi \le 180°$
- $\vec{a}, \vec{b}, \vec{c}$ is a right-hand-system

interpretation:



$\sin\varphi = \dfrac{h}{|\vec{b}|} \rightarrow h = |\vec{b}|\cdot\sin\varphi$

area of parallelogram: $A = |\vec{a}|\cdot h = |\vec{a}||\vec{b}|\cdot\sin\varphi$

$\Rightarrow A = |\vec{a}\times\vec{b}|$

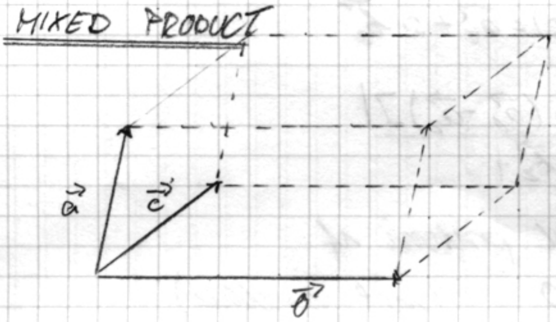
$\Rightarrow \vec{a}\times\vec{b} = \vec{0}$ iff $\vec{a}$ & $\vec{b}$ are collinear vectors

calculation:

- $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c} \qquad (\vec{b}+\vec{c})\times\vec{a} = \vec{b}\times\vec{a} + \vec{c}\times\vec{a}$

  $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$
- BUT: $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
- $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Rightarrow \vec{a}\times\vec{b} = \begin{pmatrix} a_2 b_3 - b_2 a_3 \\ -(a_1 b_3 - b_1 a_3) \\ a_1 b_2 - b_1 a_2 \end{pmatrix}$

MIXED PRODUCT



$[\vec{a}\,\vec{b}\,\vec{c}] := \vec{a}\cdot(\vec{b}\times\vec{c})$ (is a number!)

- $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$
- $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{b}\,\vec{a}\,\vec{c}]$
- $[\vec{a}\,\vec{b}\,\vec{c}] = V$ = volume of cubic element
- $[\vec{a}\,\vec{b}\,\vec{c}] = \det(\vec{a}, \vec{b}, \vec{c})$

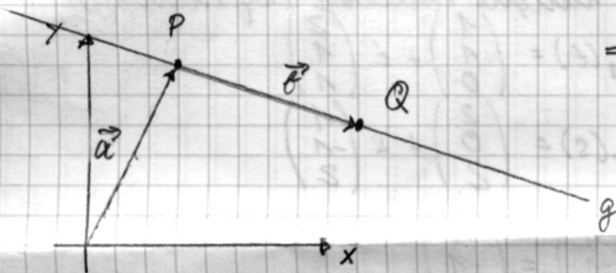
$$= \left|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

$$= a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - c_1 b_2 a_3 - a_1 c_2 b_3 - b_1 a_2 c_3$$

$$= a_1(b_2 c_3 - b_3 c_2) + a_2(b_3 c_1 - b_1 c_3) + a_3(b_1 c_2 - b_2 c_1)$$

===== application (think of $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$)

LINES

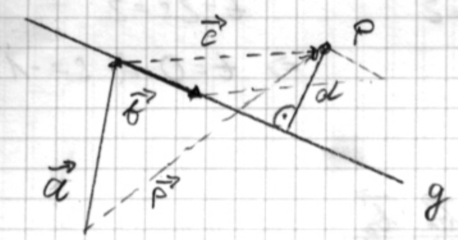


$\Rightarrow g(t) = \vec{a} + t\cdot\vec{b}$

$$g(t) = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + t\begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$$ $\rightsquigarrow$ 3 linear equations define a line in $\mathbb{R}^3$

- $g(0) = P \quad g(1) = Q$

- **DISTANCE: POINT TO LINE, $d = ?$**



$g(t) = \vec{a} + t\cdot\vec{b}$

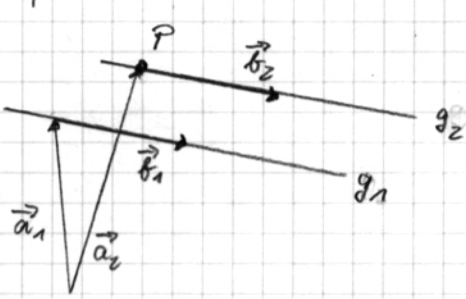
we have $\vec{c} = \vec{p} - \vec{a}$ with $\vec{p} = \overrightarrow{OP}$

$\Rightarrow d$ is the height of the parallelogram spanned by $\vec{b}$ & $\vec{c}$

$\Rightarrow |\vec{b}|\cdot d = A = |\vec{b}\times\vec{c}| = |\vec{b}\times(\vec{p}-\vec{a})|$

$$\Rightarrow d = \frac{|\vec{b}\times(\vec{p}-\vec{a})|}{|\vec{b}|}$$

- **DISTANCE: LINE TO LINE** $\rightarrow$ parallel
  $\rightarrow$ general situation
  - parallel



$g_1(t) = \vec{a}_1 + t\cdot\vec{b}_1 \qquad g_2(s) = \vec{a}_2 + s\cdot\vec{b}_2$

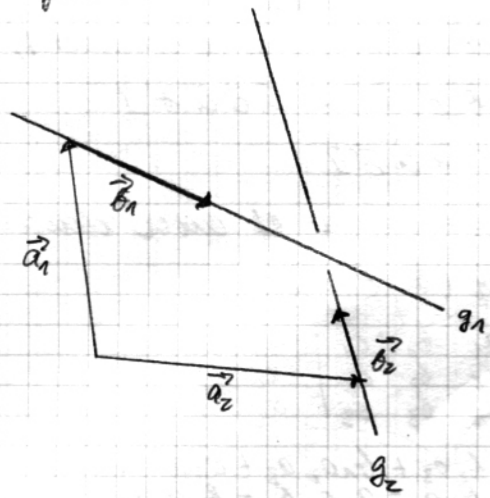
$g_1 \parallel g_2$ iff $\vec{b}_1$ & $\vec{b}_2$ colinear iff $\vec{b}_1\times\vec{b}_2 = 0$

distance $d(g_2, g_1) = d(P, g_1)$

$$\Rightarrow d = \frac{|\vec{b}_1\times(\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1|}$$

check: parallel?
then: calc dist

• general

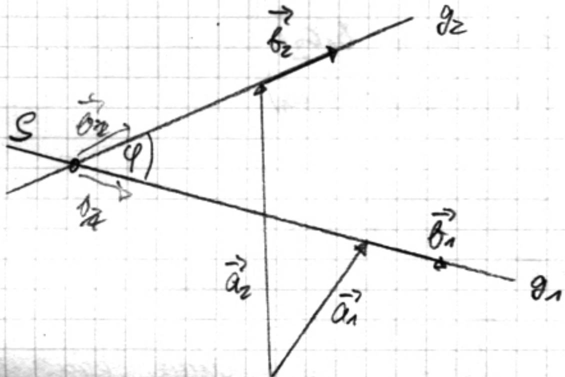


$g_1(t) = \vec{a_1} + t \cdot \vec{b_1} \quad g_2(s) = \vec{a_2} + s \cdot \vec{b_2}$

$$\Rightarrow \quad d = \frac{|[\vec{b_1}\ \vec{b_2}\ (\vec{a_1} - \vec{a_2})]|}{|\vec{b_1} \times \vec{b_2}|}$$

• lines are in general position if
- $\vec{b_1} \times \vec{b_2} \neq 0$ (not parallel)
- $[\vec{b_1}\ \vec{b_2}\ (\vec{a_1} - \vec{a_2})] \neq 0$ (no intersection)

INTESECTION OF LINES



$g_1(t) = \vec{a_1} + t \cdot \vec{b_1} \quad g_2(s) = \vec{a_2} + s \cdot \vec{b_2}$

intersection $\Rightarrow g_1(t) = g_2(s)$

$\Rightarrow \vec{a_1} + t \cdot \vec{b_1} = \vec{a_2} + s \cdot \vec{b_2}$

$\Rightarrow$ 3 equations with 2 unknowns
$\Rightarrow$ unique solution (by solving)

Example: $g_1(t) = \begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}$

$g_2(s) = \begin{pmatrix}2\\0\\2\end{pmatrix} + s\begin{pmatrix}1\\-1\\2\end{pmatrix}$

$$\Rightarrow \begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\\2\end{pmatrix} + s\begin{pmatrix}1\\-1\\2\end{pmatrix} \Rightarrow t\begin{pmatrix}2\\1\\1\end{pmatrix} - s\begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\Rightarrow \begin{aligned}2t - s &= 1\\ t + s &= -1\\ t - 2s &= 2\end{aligned} \quad \Rightarrow \quad \begin{aligned}2t - 1 &= s\\ t + s &= -1\\ t - 2s &= 2\end{aligned} \quad \Rightarrow \quad \begin{aligned}t + (2t-1) &= -1\\ t - 2s &= 2\end{aligned} \quad \Rightarrow \quad \begin{aligned}3t &= 0\\ t - 2s &= 2\end{aligned}$$

$\Rightarrow t = 0$ & $s = -1$

$$\Rightarrow S = g_1(0) = \vec{a_1} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

angle of intersection: use $\vec{b_1}$ & $\vec{b_2}$ $\Rightarrow \cos\varphi = \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}| \cdot |\vec{b_2}|}$

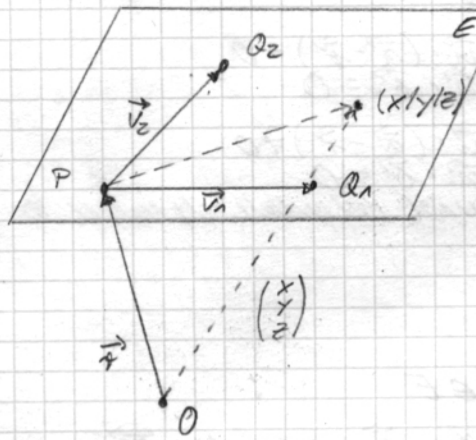
Example: $\vec{b_1} = \begin{pmatrix}2\\1\\1\end{pmatrix} \quad \vec{b_2} = \begin{pmatrix}1\\-1\\2\end{pmatrix} \Rightarrow |\vec{b_1}| = \sqrt{4+1+1} = \sqrt{6}$

$|\vec{b_2}| = \sqrt{1+1+4} = \sqrt{6}$

$\vec{b_1} \cdot \vec{b_2} = 2 - 1 + 2 = 3$

$$\Rightarrow \cos\varphi = \frac{3}{6} = \frac{1}{2} \Rightarrow \varphi = 60°$$

PLANES



equation of plane using vectors

$$\Rightarrow \boxed{E(t,s) := \vec{r} + t\cdot\vec{v_1} + s\cdot\vec{v_2}}$$

with $\vec{v_1}\times\vec{v_2} \neq 0$
i.e. not collinear

the normal $\vec{n}$ is a vector perpendicular to the plane and can be constructed out of vector product :

$$\vec{n} = \vec{v_1}\times\vec{v_2}$$

for every vector $\vec{v}$ in the plane we have

$$\vec{n}\cdot\vec{v} = 0$$

i.e $\vec{v}$ perpendicular to $\vec{n}$
$\Rightarrow$ for every point $(x/y/z)$ in the plane we have

$$\boxed{\vec{n}\cdot\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \vec{r}\right) = 0}$$

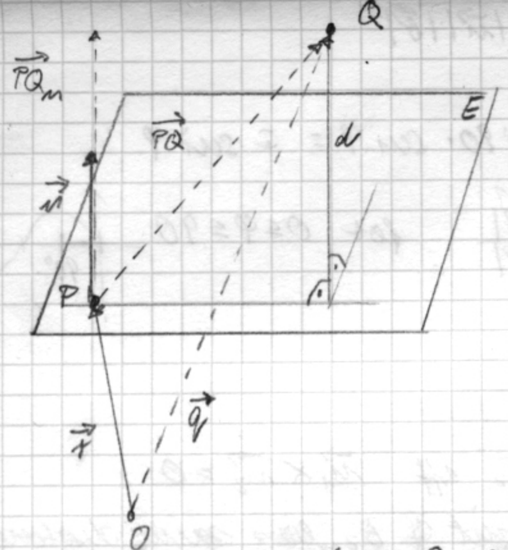
equation of plane using normal

$$\Rightarrow n_1\cdot(x - r_1) + n_2(y - r_2) + n_3(z - r_3) = 0$$

general: $\boxed{E(x,y,z) = ax + by + cz + d = 0}$

equation of plane using coordinates

- DISTANCE : POINT TO PLANE



E given by normal, that is

$$\vec{n}\cdot(\vec{v} - \vec{r}) = 0$$

we have $\overrightarrow{PQ} = \vec{q} - \vec{r}$
projection of this vector onto $\vec{n}$ has length d

$$|\overrightarrow{PQ_n}| = d$$

and for the projection we know

$$\overrightarrow{PQ_n} = \left(\frac{\vec{n}\cdot\overrightarrow{PQ}}{|\vec{n}|^2}\right)\vec{n} = \left(\frac{\vec{n}\cdot(\vec{q}-\vec{r})}{|\vec{n}|^2}\right)\vec{n}$$

$$\Rightarrow d = |\overrightarrow{PQ_n}| = \frac{|\vec{n}\cdot(\vec{q}-\vec{r})|}{|\vec{n}|}$$

if $d = 0$ then Q is in E and we have

$$0 = \frac{|\vec{n}\cdot(\vec{q}-\vec{r})|}{|\vec{n}|} \Leftrightarrow 0 = |\vec{n}\cdot(\vec{q}-\vec{r})| \Leftrightarrow 0 = \vec{n}\cdot(\vec{q}-\vec{r})$$

which is the equation of E using normal

- DISTANCE : LINE to PLANE $\Longrightarrow$ g in E / g parallel to E / g intersects E

- if $g: \vec{a}+t\vec{b}$ in $E: \vec{n}\cdot(\vec{v}-\vec{r})=0$ then $\vec{n}\cdot(\vec{a}-\vec{r})=0$ & $\vec{n}\cdot\vec{b}=0$

- if g parallel to $E: \vec{n}\cdot(\vec{v}-\vec{r})=0$ then $\vec{n}\cdot(\vec{a}-\vec{r})\neq 0$ & $\vec{n}\cdot\vec{b}=0$

  use point A given by $\vec{a}$ as point on line and calculate distance to E

  $$\Rightarrow \quad d=\frac{|\vec{n}\cdot(\vec{a}-\vec{r})|}{|\vec{n}|}$$

- if g intersects E then $\vec{n}\cdot\vec{b}\neq 0$

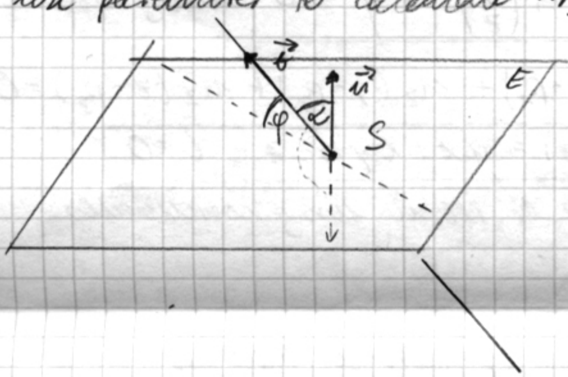
  let S be intersection point then $S\in g$ and $S\in E$

  $$\Rightarrow \quad \vec{a}+t\vec{b}=\vec{r}_S \quad \& \quad \vec{n}\cdot(\vec{r}_S-\vec{r})=0 \quad \text{with } \vec{r}_S \text{ given by } \vec{OS}$$

  $$\Rightarrow 0=\vec{n}\cdot(\vec{a}+t\vec{b}-\vec{r})=\vec{n}\cdot(\vec{a}-\vec{r}+t\vec{b})=\vec{n}\cdot(\vec{a}-\vec{r})+t(\vec{n}\cdot\vec{b})$$

  $$\Rightarrow \quad t=-\frac{\vec{n}\cdot(\vec{a}-\vec{r})}{\vec{n}\cdot\vec{b}}=\frac{\vec{n}\cdot(\vec{r}-\vec{a})}{\vec{n}\cdot\vec{b}}$$

  use parameter to calculate $\vec{r}_S=\vec{a}+\left(\frac{\vec{n}\cdot(\vec{r}-\vec{a})}{\vec{n}\cdot\vec{b}}\right)\cdot\vec{b}$



for angle of intersection $0\le\varphi\le 90^\circ$ we have

$$\alpha=90^\circ-\varphi \quad \text{or} \quad \alpha=90^\circ+\varphi$$

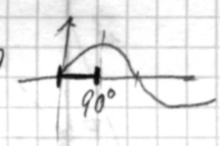
(depending on orientation of $\vec{n}$). we know

$$\cos\alpha=\frac{\vec{n}\cdot\vec{b}}{|\vec{n}|\cdot|\vec{b}|}$$

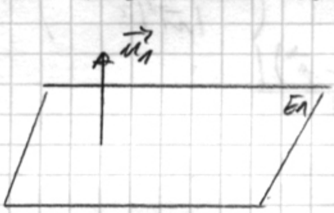
for $\alpha=90^\circ\pm\varphi$ follows

$$\cos\alpha=\cos(90^\circ\pm\varphi)=\underbrace{\cos 90^\circ}_{=0}\cdot\cos\varphi\mp\underbrace{\sin 90^\circ}_{=1}\cdot\sin\varphi=\mp\sin\varphi$$

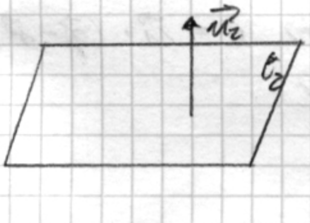
$$\Rightarrow \quad \sin\varphi=\mp\frac{\vec{n}\cdot\vec{b}}{|\vec{n}|\cdot|\vec{b}|} \quad\Rightarrow\quad \sin\varphi=\frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|\cdot|\vec{b}|} \quad \text{for } 0\le\varphi\le 90$$



- DISTANCE : PLANE to PLANE $\Longrightarrow$ $E_1=E_2$ / $E_1 \parallel E_2$ / E_1 intersects E_2



$$\vec{n}_1\cdot(\vec{v}-\vec{r}_1)=0$$



$$\vec{n}_2\cdot(\vec{v}-\vec{r}_2)=0$$

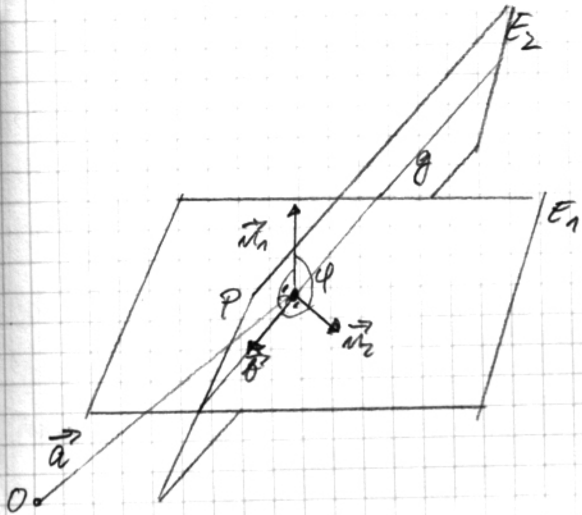
$E_1 \parallel E_2$ iff $\vec{n}_1\times\vec{n}_2=0$

every point of E_2 has same distance to E_1

use $\vec{r}_2$ and calculate d

$$\Rightarrow \quad d=\frac{|\vec{n}_1\cdot(\vec{r}_2-\vec{r}_1)|}{|\vec{n}_1|}$$

if $d=0$ then $E_1=E_2$



E_2: $\vec{n}_2 \cdot (\vec{v} - \vec{r}_2) = 0$

E_1: $\vec{n}_1 \cdot (\vec{v} - \vec{r}_1) = 0$

E_1 intersects E_2 iff $\vec{n}_1 \times \vec{n}_2 \neq 0$

for intersection line $g: \vec{a} + t\vec{b}$ we have

$\vec{b} \perp \vec{n}_1$ & $\vec{b} \perp \vec{n}_2$

$\Rightarrow \quad \vec{b} := \vec{n}_1 \times \vec{n}_2$

for $\vec{a}$ we know $\vec{n}_1 \cdot (\vec{a} - \vec{r}_1) = 0$ & $\vec{n}_2 \cdot (\vec{a} - \vec{r}_2) = 0$

$\Rightarrow$ linear system with 2 equations and 3 unknowns
define $a_1 = 0$ and calculate a_2, a_3 out of equations

for angle of intersection use normal

$$\Rightarrow \quad \cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$$